Math 245C Lecture 8 Notes

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1 Bounds on Integral Operators (cont.)

1.1 Proof of the weak and strong type properties

Last time, we were proving the following theorem:

Theorem 1.1. *Let $1 \leq p < \infty$ and $c > 0$. Assume that $[K(x, \cdot)]_q \leq C$ for μ-a.e. $x \in X$ and $[K(\cdot, w)]_w \leq C$ for ν-a.e. $y \in Y$.*

1. *If $1 \leq p < \infty$, $L^p(\nu) \subseteq \mathcal{F}$.*
2. *If $1 < p < r < \infty$, then there exist $B_1 > 0$ and $B_p > 0$ such that $[Tf]_q \leq B_1 \|f\|_1$ and $\|Tf\|_r \leq CB_p\|f\|_p$, which means T is weak type $(1, q)$ and strong type (p, r), provided that $1/r + 1 = 1/p + 1/q$.*

Proof. It remains to show the second conclusion. We have fixed f such that $\|f\|_p = 1$. We have already obtained the following useful identities:

$$\int_X |K_1(x, y)| \, d\nu(y), \int_K |K_1(x, y)| \, d\nu(x) \leq C \frac{A^{1-q}}{q-1}$$

$$\|T_2 f\| \leq A^{q/r} \left(\frac{cr}{q} \right)^{1/p}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

We chose A such that $A^{q/r}(cr/q)^{1/p'} = \alpha/2$. These give us

$$\lambda_{T_2 f}(\alpha/2) = 0.$$

So

$$\lambda_{Tf}(\alpha) \leq \lambda_{T_1 f}(\alpha/2) + \lambda_{T_2 f}(\alpha/2) = \lambda_{T_1 f}(\alpha/2).$$

Now apply the following observation to $h = T_1 f$:

$$\int |h|^p \, d\nu \geq \int_{\{|h| > \alpha/2\}} |h|^p \, d\nu \geq \left(\frac{\alpha}{2} \right)^p \lambda_h(\alpha/2) \implies \lambda_h(\alpha/2) \leq \left(\frac{\alpha}{2} \right)^{-p} \|h\|_p^p.$$

We get

$$\begin{aligned}
\lambda_{Tf}(\alpha) &\leq \left(\frac{\alpha}{2}\right)^{-p} \|T_1 f\|_p^p \\
&\leq \left(\frac{\alpha}{2}\right)^{-p} \left(c\frac{A^{1-q}}{q-1}\right)^p \\
&= \left(\frac{\alpha}{2}\right)^{-p} \left(\frac{c}{q-1}\alpha^{r/q} \left[\frac{1}{2}\left(\frac{q}{cr}\right)^{1/p'}\right]^{r/q}\right)^{(1-q)p} \\
&= \alpha^{-p+r/q(1-q)p} C(q,p).
\end{aligned}$$

Now we note that

$$-p + r/q(1-q)p = p((1/q-1)-1) = p(r(1/r-1/p)-1) = -r/p.$$

So, by homogeneity,

$$\alpha^r \lambda_{Tf}(\alpha) \leq C(q,p)\|f\|_p^r.$$

In particular, when $p=1$, then $r=q$, and we get that

$$\alpha^q \lambda_{Tf}(\alpha) \leq C(q,1)\|f\|_q^q.$$

That is, T is weak type $(1,q)$.

We next need to find (p_1, r_1) such that T is weak type (p_1, r_1), where $q \geq 1$ and $p_1 \leq r_1$. Choose $p_1 \in (p, \infty)$ close enough to p. Let $t \in (0,1)$ be such that

$$\frac{1}{p} = \frac{1-t}{1} + \frac{t}{p_1}.$$

Define r_1 by

$$\frac{1}{r} = \frac{1-t}{q} + \frac{t}{r_1}.$$

Since p is close to p_1, r is close to r_1. By the definition of r_1,$r_1 < r$. We have

$$\alpha^{r_1} \lambda_{Tf}(\alpha) \leq C(q,p_1)\|f\|_{p_1}^{r_1}.$$

This means that T is weak type (p_1, r_1). Since T is also weak type $(1,q)$ the Marcinkiewicz interpolation theorem gives us that T is strong type (p,r). □

1.2 Preliminaries for Fourier analysis

Notation: We will assume that $n \geq 1$ is a natural number. If $x = (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$, then

$$x \cdot y = \sum_{i=1}^n x_i y_i, \qquad \|x\|^2 = x \cdot x.$$

If $\alpha \in \mathbb{N}^n$, then

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \qquad \alpha! = \prod_{i=1}^{n} (\alpha_i!).$$

We will also write

$$x^\alpha = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}), \qquad \partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}}.$$

With this notation, the Taylor expansion is

$$f(x) = \sum_{|\alpha| \le k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x_0)(x - x_0)^\alpha + R_k(x), \qquad \text{where } \lim_{x \ tox_0} \frac{R_k(x)}{|x - x_0|^k} = 0.$$

Define

$$\eta(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0. \end{cases}$$

We have $\eta \in C^\infty(\mathbb{R})$, as

$$\frac{x^n}{e^x} \xrightarrow{x \to \infty} 0$$

for each n. By induction, we can show that $\eta^{(k)}(0) = 0$ for all $k \ge 1$.

For $x \in \mathbb{R}^n$, set

$$\rho(x) = \eta(1 - \|x\|^2) = \begin{cases} e^{1/(\|x\|^2 - 1)} & \|x\| < 1 \\ 0 & \|x\| > 1. \end{cases}$$

Then $\mathrm{supp}(\rho) = \overline{B_1(0)}$, $\rho \in C^\infty$, $\rho > 0$, and $\rho(-x) = x$.